Geometry of D-branes for general $N = (2,2)$ sigma models

Maxim Zabzine[1]

Lab. de Physique Théorique et Hautes Energies
Université Pierre et Marie Curie, Paris VI
4 Place Jussieu, 75252 Paris Cedex 05, France

Institut Mittag-Leffler
Auravägen 17, S-182 62 Djursholm, Sweden

ABSTRACT

We give a world-sheet description of D-brane in terms of gluing conditions on $T\mathcal{M} \oplus T^*\mathcal{M}$. Using the notion of generalized Kähler geometry we show that A- and B-types D-branes for the general $N = (2,2)$ supersymmetric sigma model (including a non-trivial NS-flux) correspond to the (twisted) generalized complex submanifolds with respect to the different (twisted) generalized complex structures however.

---

[1]e-mail address: zabzine@lpthe.jussieu.fr

1   Introduction

The aim of this note is to give a geometrical description of A- and B- types of branes for the *general* $N = (2,2)$ sigma model. These branes correspond to a maximally supersymmetric boundary conditions for the $N = (2,2)$ sigma model, i.e. they preserve the half of the bulk world-sheet supersymmetry.

Despite the fact that these sigma models were known for twenty years [10] the appropriate geometrical language has been missing. Recently Hitchin introduced the notion of generalized complex geometry [12] which provides a natural set up for the relevant geometry. Later the geometry of the *general* $N = (2,2)$ sigma model has been formalized in Gualtieri's thesis [11] and now it goes under the name of (twisted) generalized Kähler geometry. The algebraic set-up for the generalized Kähler geometry has been discussed previously in physics literatures [13].

This paper is inspired by Kapustin's work [15] and it presents the generalization of some of his proposals. The idea is to analyze the geometry of a sigma model and its D-branes not in terms of tangent space $T\mathcal{M}$ but in terms of tangent and cotangent spaces $T\mathcal{M} \oplus T^*\mathcal{M}$. We will show that the appropriate D-branes have a simple description in terms of $T\mathcal{M} \oplus T^*\mathcal{M}$.

The article is organized as follows. In Section 2 we review the geometry of $N = (2,2)$ general sigma model. In particular we briefly sketch the description of this geometry in terms of $T\mathcal{M} \oplus T^*\mathcal{M}$, a generalized Kähler geometry recently introduced by Gualtieri [11]. Section 3 deals with the description of $N = 1$ boundary conditions in terms of gluing map defined over $T\mathcal{M} \oplus T^*\mathcal{M}$. In the following Section 4 we discuss the geometrical interpretation of A- and B-types branes for the general $N = 2$ sigma model. We present some examples of $N = (2,2)$ branes. Finally, in Section 5 we give a summary of the paper with a discussion of open problems.

2   Generalized Kähler geometry

In this section we review the results on the general $N = (2,2)$ supersymmetric sigma model discovered in [10]. In particular we discuss the bi-Poisson property of this geometry [19]. Finally we sketch the notion of (twisted) generalized complex and generalized Kähler geometries [11]. This Section will allow us to introduce the notation and some relevant concepts.

Let us start from the general $N = (1,1)$ sigma model (without boundaries) which is

written in $N = (1, 1)$ superfields

$$S = \int d^2\sigma \, d^2\theta \; D_+\Phi^\mu D_-\Phi^\nu (g_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)), \tag{2.1}$$

where $H = dB$ on some patch. Although B is used to write the action (2.1) down the theory depends only on the closed three form H. Thus a manifold $(\mathcal{M}, g, H)$ with a metric and a closed three form admits $N = (1, 1)$ supersymmetric sigma model.

We look for additional supersymmetry transformations of the form

$$\delta\Phi^\mu = \epsilon^+ D_+\Phi^\nu J^\mu_{+\nu}(\Phi) + \epsilon^- D_-\Phi^\nu J^\mu_{-\nu}(\Phi). \tag{2.2}$$

Classically the ansatz (2.2) is unique for dimensional reasons. It turns out that the action (2.1) is invariant under the transformations (2.2) provided that

$$g_{\nu\mu} J^\mu_{\pm\rho} = -g_{\rho\mu} J^\mu_{\pm\nu} \tag{2.3}$$

and

$$\nabla^{(\pm)}_\rho J^\mu_{\pm\nu} \equiv J^\mu_{\pm\nu,\rho} + \Gamma^{\pm\mu}_{\rho\sigma} J^\sigma_{\pm\nu} - \Gamma^{\pm\sigma}_{\rho\nu} J^\mu_{\pm\sigma} = 0, \tag{2.4}$$

where one defines two affine connections

$$\Gamma^{\pm\mu}_{\rho\nu} = \Gamma^\mu_{\rho\nu} \pm g^{\mu\sigma} H_{\sigma\rho\nu}. \tag{2.5}$$

Next we have to require the standard on-shell $N = (2, 2)$ supersymmetry algebra, i.e. the manifest supersymmetry transformations and the second supersymmetry transformations (2.2) commute and the commutator of two second supersymmetry transformations (2.2) gives a translation. Thus the supersymmetry algebra requires that $J_\pm$ correspond to two complex structures.

This is the full description of the most general N=(2,2) sigma model (2.1). Thus the target manifold should be a bihermitian complex manifold (i.e., there are two complex structures and a metric is Hermitian with respect to both) and the two complex structures should be covariantly constant, with respect to the different connections however. Thus a manifold with this data $(\mathcal{M}, g, J_+, J_-, H)$ admits $N = (2, 2)$ supersymmetric sigma model. If $H = 0$ then J_+ and J_- give rise to the standard Kähler structures which maybe different, $J_+ \neq J_-$.

The above geometry can be reformulated in different but an equivalent language. For example, we can describe the geometry as being bi-Poisson [19]. Introducing the two-forms $\omega_\pm = gJ_\pm$ we can show that $(\omega_+^{-1} \pm \omega_-^{-1})$ are Poisson tensors. Infact the bi-Poisson property is equivalent to the conditions (2.4). However the property $dH = 0$ is lacking in this description.

Recently Hitchin [12] has introduced the notion of generalized complex geometry. In Hitchin's construction $T\mathcal{M}$ is replaced by $T\mathcal{M} \oplus T^*\mathcal{M}$ and the Lie bracket is replaced by

the appropriate bracket on $T\mathcal{M} \oplus T^*\mathcal{M}$, the so called Courant bracket. Thus a generalized complex structure is an almost complex structure $\mathcal{J}$ on $T\mathcal{M} \oplus T^*\mathcal{M}$ whose $+i$-eigenbundle is Courant involutive. Using the language of generalized complex geometry Gualtieri reformulated the geometry of $N = (2, 2)$ sigma models and it goes under the name of generalized Kähler geometry. This reformulation is essential for our further discussion. Below we will briefly review the relevant concepts of this construction. A detailed presentation of generalized complex and generalized Kähler geometries can be found in Gualtieri's thesis [11].

We start by providing the definition of generalized complex structure. On $T\mathcal{M} \oplus T^*\mathcal{M}$ there is a natural indefinite metric defined by $(X + \xi, X + \xi) = i_X\xi$. In the coordinate basis (∂_μ, dx^μ) we can write this metric as follows

$$\mathcal{I} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}. \tag{2.6}$$

An almost generalized complex structure is a map $\mathcal{J} : T\mathcal{M} \oplus T^*\mathcal{M} \to T\mathcal{M} \oplus T^*\mathcal{M}$ such that $\mathcal{J}^2 = -1_{2d}$ and that $\mathcal{I}$ is hermitian with respect to $\mathcal{J}$, $\mathcal{J}^t\mathcal{I}\mathcal{J} = \mathcal{I}$. On $T\mathcal{M} \oplus T^*\mathcal{M}$ there is a Courant bracket which is defined as follows

$$[X + \xi, Y + \eta]_c = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi), \tag{2.7}$$

where $[\ ,\ ]$ is a Lie bracket on $T\mathcal{M}$. This bracket is skew-symmetric but in general does not satisfy the Jacobi identity. However if there is a sub-bundle $L \subset T\mathcal{M} \oplus T^*\mathcal{M}$ which is involutive (closed under the Courant bracket) and isotropic with respect to $\mathcal{I}$ then the Courant bracket on the sections of L does satisfy the Jacobi identity. This is a reason for imposing hermiticity of $\mathcal{I}$ with respect to $\mathcal{J}$. We can construct the projectors on $T\mathcal{M} \oplus T^*\mathcal{M}$

$$\Pi_\pm = \frac{1}{2}(I \pm i\mathcal{J}), \tag{2.8}$$

the almost generalized complex structure $\mathcal{J}$ is integrable if

$$\Pi_\mp[\Pi_\pm(X + \xi), \Pi_\pm(Y + \eta)]_c = 0, \tag{2.9}$$

for any $(X + \xi), (Y + \eta) \in T\mathcal{M} \oplus T^*\mathcal{M}$. The Courant bracket on $T\mathcal{M} \oplus T^*\mathcal{M}$ can be twisted by a closed three form H. Namely given a closed three form H one can define another bracket on $T\mathcal{M} \oplus T^*\mathcal{M}$ by

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta]_c + i_X i_Y H. \tag{2.10}$$

This bracket has similar properties to the Courant bracket. Again if a sub-bundle $L \subset T \oplus T^*$ is closed under the twisted Courant bracket and isotropic with respect to $\mathcal{I}$, then the Courant bracket on the sections of L does satisfy the Jacobi identity. Thus in the integrability

4

condition (2.9) the Courant bracket $[\ ,\ ]_c$ can be replaced by the new twisted Courant bracket $[\ ,\ ]_H$. In local coordinates the integrability conditions have been worked out in [18].

One important feature of the (twisted) Courant bracket is that, unlike the Lie bracket, this bracket has a nontrivial automorphism defined by a closed two-form b. Namely if we define the following action on $T\mathcal{M} \oplus T^*\mathcal{M}$

$$e^b(X + \xi) = X + \xi + i_X b. \tag{2.11}$$

with an arbitrary two-form b. To the transformation (2.11) we refer as a b-transform. Under the b-transform the (twisted) Courant bracket behaves as

$$[e^b(X + \xi), e^b(Y + \eta)]_H = e^b[X + \xi, Y + \eta]_{H+db}. \tag{2.12}$$

Thus if $db = 0$ the action (2.11) is automorphism of (twisted) Courant bracket.

Another important property is that under the natural projection $p : T\mathcal{M} \oplus T^*\mathcal{M} \to T\mathcal{M}$ the (twisted) Courant bracket satisfies the following condition

$$p[X + \xi, Y + \eta]_H = [p(X + \xi), p(Y + \eta)]_H = [X, Y]. \tag{2.13}$$

Therefore if a sub-bundle L of $T\mathcal{M} \oplus T^*\mathcal{M}$ is closed under the (twisted) Courant bracket then its image $p(L)$ is closed under the Lie bracket and thus corresponds to a foliation.

Next we present the definition of generalized Kähler manifold [11]. The (twisted) generalized Kähler structure is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting (twisted) generalized complex structures such that $G = -\mathcal{J}_1 \mathcal{J}_2$ is a positive definite metric on $T\mathcal{M} \oplus T^*\mathcal{M}$. As it has been shown in [11] the generalized Kähler geometry $(\mathcal{M}, \mathcal{J}_1, \mathcal{J}_2)$ is completely equivalent to the geometry of $N = (2, 2)$ sigma models $(\mathcal{M}, g, J_+, J_-, H)$ [10].

The correspondence works as follows: Starting from the Gates-Hull-Roček geometry $(\mathcal{M}, g, J_+, J_-, H)$ we can construct two $\tilde{H}$-twisted commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$

$$\mathcal{J}_{1/2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^t \pm J_-^t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{2.14}$$

where b is two-form (which maybe zero) and $H = \tilde{H} + db$. The generalized metric $G = -\mathcal{J}_1 \mathcal{J}_2$ is given

$$G = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}. \tag{2.15}$$

The opposite is also true: Starting from a (twisted) generalized Kähler geometry $(\mathcal{M}, \mathcal{J}_1, \mathcal{J}_2)$ we can recover the complex structures $J_\pm$ with the right properties.

3 D-branes in $O(d,d)$ formalism

Before discussing D-branes (vs boundary conditions) for $N = (2,2)$ sigma model we consider the $N = 1$ superconformal boundary conditions reformulated in terms of $T\mathcal{M} \oplus T^*\mathcal{M}$.

The boundary conditions for $N = (1,1)$ models have been discussed extensively during last years both at classical and quantum levels. Here we are interested in the geometrical interpretation of D-branes and our discussion is entirely within the classical (semi-classical) sigma model. On the boundary the following gluing conditions are imposed

$$\psi_-^\mu = \eta R_\nu^\mu(X)\psi_+^\nu, \tag{3.16}$$

where $\eta = \pm 1$ corresponds to the choice of spin structure. The expression (3.16) is most general local ansatz for the fermionic boundary conditions compatible with dimensional analysis. The bosonic counterpart of (3.16) and the restrictions on gluing matrix R can be derived by requiring $N = 1$ superconformal symmetry. This problem has been worked out in great details in [1, 2], where the differential conditions have been also considered.

Let us summarize the relevant results from [1, 2]. The gluing matrix $R : T\mathcal{M} \to T\mathcal{M}$ has an interpretation in terms of submanifold with an extra data. We defined a (maximal) projector Q (i.e., $Q^2 = Q$) such that $RQ = QR = -Q$. The complementary projector $\pi = 1_d - Q$ defines the integrable distribution. Thus there is a maximal integral submanifold $\mathcal{D}$ which corresponds to π. There exists a two-form $F \in \Omega^2(\mathcal{D})$ on $\mathcal{D}$ such that $dF = H|_\mathcal{D}$. Along $\mathcal{D}$ the gluing matrix is defined as follows

$$\pi^t(g - F)\pi = \pi^t(g + F)\pi R. \tag{3.17}$$

As well the gluing matrix respects the metric, $R^t g R = g$ and therefore $\mathcal{D}$ is a Riemannian submanifold. For further use let us introduce the following object $r = \pi - Q$. It follows from above properties that $r^2 = 1_d$ and $r^t g r = g$.

Now our goal is to reformulate above description in $O(d,d)$ covariant terms. Following the idea from [15] we can introduce

$$\psi^\mu = \frac{1}{2}(\eta\psi_+^\mu + \psi_-^\mu), \qquad \rho_\mu = \frac{1}{2}g_{\mu\nu}(\eta\psi_+^\nu - \psi_-^\nu) \tag{3.18}$$

where ψ takes values in the pull-back of $T\mathcal{M}$ and ρ in the pull-back of $T^*\mathcal{M}$. Together they combine to $\mathbf{\Psi} = (\psi, \rho) \in T\mathcal{M} \oplus T^*\mathcal{M}$. We can introduce the new gluing matrix $\mathcal{R}$ defined on $T\mathcal{M} \oplus T^*\mathcal{M}$

$$\mathcal{R} = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & -r^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix} = \begin{pmatrix} r & 0 \\ Fr + r^t F & -r^t \end{pmatrix}. \tag{3.19}$$

The boundary conditions (3.16) are written now as follows

$$\mathcal{R}\boldsymbol{\Psi} = \boldsymbol{\Psi}. \tag{3.20}$$

In this boundary condition two form F along $\mathcal{D}$ matters only. By explicit calculation one can show that the condition (3.20) with (3.19) encodes completely the algebraic part of the boundary conditions (3.16).

Let us summarize the properties of $\mathcal{R}$ given by (3.19): $\mathcal{R}$ respects the natural pairing $\mathcal{I}$ on $T\mathcal{M} \oplus T^*\mathcal{M}$, $\mathcal{R}^t \mathcal{I} \mathcal{R} = -\mathcal{I}$ (i.e., $\mathcal{R} \in so(d,d)$). Also $\mathcal{R}^2 = 1_{2d}$ and thus we can defined the projector operator $\frac{1}{2}(1_{2d} + \mathcal{R})$. This projector operator defines a real, maximal isotropic sub-bundle of $T\mathcal{M} \oplus T^*\mathcal{M}|_{\mathcal{D}}$:

$$\tau_{\mathcal{D}}^F = \{\frac{1}{2}(1_{2d} + \mathcal{R})(X + \xi) = (X + \xi): \ \ X + \xi \in T\mathcal{D} \oplus T^*\mathcal{M}|_{\mathcal{D}}, \ \xi|_{\mathcal{D}} = i_X F\}. \tag{3.21}$$

This coincides with the definition of generalized tangent bundle of generalized submanifold given in [11]. Introducing the generalized metric G as in (2.15) (but instead of b use F) we can rewrite the property $R^t g R = g$ as follows

$$GR + RG = 0. \tag{3.22}$$

Indeed the differential conditions (i.e., the integrability of π and $dF = H|_{\mathcal{D}}$) can be encoded in the requirement that the sub-bundle (3.21) is involutive with respect to H-twisted Courant bracket defined on the sections of $T\mathcal{M} \oplus T^*\mathcal{M}$,

$$(1_{2d} - \mathcal{R})[(1_{2d} + \mathcal{R})(X + \xi), (1_{2d} + \mathcal{R})(Y + \eta)]_H = 0. \tag{3.23}$$

Thus a D-brane (vs $N = 1$ superconformal boundary condition) can be naturally described in terms of $T\mathcal{M} \oplus T^*\mathcal{M}$.

In this context we would like to make the following speculative observation. If $\mathcal{R}$ is lower triangular matrix with the properties $\mathcal{R}^t \mathcal{I} \mathcal{R} = -\mathcal{I}$ and $\mathcal{R}^2 = 1_{2d}$ then it has a form (3.19) and we recover the standard world-sheet description of D-brane. There is no need for the object $\mathcal{R}$ to be globally defined on whole $\mathcal{M}$. It is tempting to drop the condition for $\mathcal{R}$ to be a lower triangular. If we do this then D-brane corresponds to a (local) involutive maximal isotropic sub-bundle of $T\mathcal{M} \oplus T^*\mathcal{M}$. With this definition we lose a standard geometric interpretation of D-brane as a submanifold. However due to the property (2.13) D-brane corresponds to a foliation on $\mathcal{M}$ which maybe singular. Infact at bosonic level this is allowed. Namely the conformality condition at the boundary is given by

$$T_{++} - T_{--} = 4\partial_1 X^\mu g_{\mu\nu} \partial_0 X^\nu = 2 \begin{pmatrix} \partial_0 X & g\partial_1 X \end{pmatrix} \mathcal{I} \begin{pmatrix} \partial_0 X \\ g\partial_1 X \end{pmatrix} = 0 \tag{3.24}$$

7

and thus $(\partial_0 X \ g\partial_1 X)$ belongs to a maximal isotropic subspace of $T\mathcal{M} \oplus T^*\mathcal{M}$. These non-geometrical branes require more study and we hope to come back to the subject elsewhere.

In next Section using the proposed description of $N = 1$ branes we study $N = 2$ branes. For the sake of clarity let us set $\eta = 1$ for the rest of the paper.

4   Geometry of N=2 D-branes

In this Section we describe D-branes for the general $N = (2, 2)$ sigma model. The formal description of these branes has been presented in [16]. However the geometrical interpretation has been lacking, except for Kähler case [20]. Using the standard gluing matrix (3.16) the preservation of $N = 2$ supersymmetry in presence of the boundary implies

$$J_- R = \pm R J_+ \tag{4.25}$$

or alternatively

$$R^t \omega_- R = \pm \omega_+, \tag{4.26}$$

where $\pm$ correspond to different ways of gluing left and right $U(1)$ (super)currents. We call the case with $+$ a B-type and with $-$ an A-type branes. When $J_+ = J_-$ we recover the Kähler case with the standard terminology introduced in [20].

Our goal is to use the observations from the previous Section and reformulate the condition (4.25) in terms of $T\mathcal{M} \oplus T^*\mathcal{M}$. Before doing this we can immediately observe one very important property of the branes: they are coisotropic manifolds with respect to different Poisson structures. Namely from (4.26) it follows that $Q^t(\omega_+^{-1} \mp \omega_-^{-1})Q = 0$ where Q is a normal projector to a brane. Thus for B-type brane we have $(\omega_+^{-1} - \omega_-^{-1})N^*\mathcal{D} \subset T\mathcal{D}$, where $N^*\mathcal{D}$ is a conormal bundle of $\mathcal{D}$. Therefore the B-type branes are coisotropic submanifolds with respect to Poisson structure $(\omega_+^{-1} - \omega_-^{-1})$. Correspondingly the A-type branes are coisotropic submanifolds with respect to Poisson structure $(\omega_+^{-1} + \omega_-^{-1})$. The distribution $(\omega_+^{-1} \mp \omega_-^{-1})N^*\mathcal{D}$ on the coisotropic submanifold $\mathcal{D}$ is involutive and the corresponding foliation is called the characteristic foliation. It is interesting to point out that the D-branes for the Poisson sigma model are also coisotropic submanifolds with respect to a Poisson structure [7] (for related discussion see also [6]).

The conditions (4.25) and (4.26) can be derived by looking at the appropriate gluing conditions for the $U(1)$ currents

$$j_+ \mp j_- = \psi_+^\mu \omega_{+\mu\nu} \psi_+^\nu \mp \psi_-^\mu \omega_{-\mu\nu} \psi_-^\nu = 0. \tag{4.27}$$

It is useful to rewrite the linear combinations of $U(1)$ currents as follows

$$j_+ - j_- = 2\mathbf{\Psi}^t \mathcal{I} \mathcal{J}_1 \mathbf{\Psi}, \qquad j_+ + j_- = 2\mathbf{\Psi}^t \mathcal{I} \mathcal{J}_2 \mathbf{\Psi} \tag{4.28}$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are H-twisted commuting generalized complex structures defined in (2.14) with $b = 0$. Combining the boundary condition (3.20) with (4.27) and (4.28) we arrive on the following description of B- and A-types of branes: For B-type brane $\mathcal{R}$ and $\mathcal{J}_1$ are commuting

$$\mathcal{R}\mathcal{J}_1 = \mathcal{J}_1\mathcal{R}, \tag{4.29}$$

and thus sub-bundle $\tau_\mathcal{D}^F$ is stable under $\mathcal{J}_1$. Therefore the sub-bundle $\tau_\mathcal{D}^F$ must decompose into $\pm i$ egeinbundles of $\mathcal{J}_1$

$$\tau_\mathcal{D}^F = \tau_\mathcal{D}^{F+} \oplus \tau_\mathcal{D}^{F-}. \tag{4.30}$$

Since $\tau_\mathcal{D}^F$ is closed under the (twisted) Courant bracket (see the previous Section) then each of $\tau_\mathcal{D}^{F\pm}$ is closed under the (twisted) Courant bracket. The distributions $p(\tau_\mathcal{D}^{F\pm})$ are involutive and give rise to the corresponding foliations on $\mathcal{D}$. For A-type brane $\mathcal{R}$ and $\mathcal{J}_2$ are commuting

$$\mathcal{R}\mathcal{J}_2 = \mathcal{J}_2\mathcal{R} \tag{4.31}$$

and thus sub-bundle $\tau_\mathcal{D}^F$ is stable under $\mathcal{J}_2$. Again as before $\tau_\mathcal{D}^F$ can be decomposed into $\pm i$ involutive eigenbundles of $\mathcal{J}_2$. Thus A- and B-types branes are both (twisted) generalized complex submanifolds (as defined in [11])[2] however with respect to different generalized complex structure.

Above we gave the geometric description of a generic B- and A-types branes for a general $N = (2, 2)$ sigma model. In the remaining part of this Section we discuss the special cases and examples of these branes. Let us consider the case when H is exact, $H = db$ and define the generalized Kähler geometry as in (2.14) and (2.15). Now we can consider a D-brane $\mathcal{R}$ as defined in (3.19) with $F = b$. Combining the relations (4.29), (4.31) with (3.22) we end up with following description

$$\mathcal{R}\mathcal{J}_1 = \pm\mathcal{J}_1\mathcal{R}, \qquad \mathcal{R}\mathcal{J}_2 = \mp\mathcal{J}_2\mathcal{R} \tag{4.32}$$

correspondingly for B- and A-types of branes. These branes correspond to a special case when $J_-r = \pm rJ_+$.

To illustrate the general discussion let us consider the well-known case of D-brane on a Kähler manifold [20]. In this situation $J_+ = J_- = J$ and the corresponding generalized complex structures are given by

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \qquad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{4.33}$$

---

[2]The related discussion of submanifolds of generalized complex manifolds is given in [4]. However we follow the definitions from [11].

The B-type branes (4.29) correspond to the Kähler submanifolds $\mathcal{D}$ with F being closed $(1,1)$ form with respect to the complex structure J. The A-type branes (4.31) correspond to the Lagrangian submanifolds and "coisotropic A-branes" discovered in [14] (see also the related discussion in [16]). For the special case (4.32) B-branes correspond to the Kähler submanifolds with $F = 0$ and A-branes to the Lagrangian submanifolds.

The example with a non-trivial H can be given by the group manifolds case. Let us consider a group manifold $\mathcal{G}$ with a Lie algebra $\mathbf{g}$. If $\mathbf{g}$ admits a complex version of Manin triple $(\mathbf{g}, \mathbf{g}_-, \mathbf{g}_+)$ then $\mathcal{G}$ has a twisted generalized Kähler geometry (vs $N = (2,2)$ WZW model). Namely a Lie algebra $\mathbf{g}$ should be equipped with symmetric ad-invariant non-degenerate bilinear form $< \, , \, >$ and $\mathbf{g}$ can be decomposed into a pair of maximally isotropic subalgebras $\mathbf{g}_-$ and $\mathbf{g}_+$ as a vector space. The closed three form H is given by $H(X,Y,Z) = < [X,Y], Z >$, for further details of construction we refer to [17]. In the same work [17] the special class of $N = 2$ D-branes related to an automorphism of the affine algebras has been discussed. An example of B-type brane is given by the trivial Lie algebra automorphism. The underlying geometry of these branes has been studied in [3] and a trivial automorphism corresponds to a conjugacy classes $\mathcal{D}$ on $\mathcal{G}$. Each conjugacy class admits a two-form F such that $dF = H|_{\mathcal{D}}$. Thus the conjugacy classes of $\mathcal{G}$ are the examples of twisted generalized complex submanifolds for $\mathcal{J}_1$, the twisted generalized complex geometry. Using the generalization of these ideas and the notion of twisted conjuagacy classes [8, 21] it is not hard to construct more examples of the (twisted) generalized complex submanifolds.

5   Summary and discussion

The geometry of the general $N = (2,2)$ sigma model is described in terms of two commuting (twisted) generalized complex structures, $\mathcal{J}_1$ and $\mathcal{J}_2$. We have shown that B-branes correspond to a (twisted) generalized complex submanifolds with respect to $\mathcal{J}_1$ and A-branes to a (twisted) generalized complex submanifolds with respect to $\mathcal{J}_2$. The geometry of these branes in terms of tangent and cotangent bundles looks simple. However the intrinsic geometry of the branes can be quite involved (e.g., there are the foliations on the world-volume, see the previous Section) and presumably depends on the details of (twisted) generalized complex geometry. Under the mirror symmetry automorphism of the world-sheet theory, the (twisted) generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ get interchanged. Thus B- and A-type branes are interchanged under the mirror map as expected.

The main lesson of this paper is that both $N = 1$ and $N = 2$ branes naturally described in terms of $T\mathcal{M} \oplus T^*\mathcal{M}$. However in this language it is tempting to extend the geometrical description of D-branes from a submanifold to a foliation (possibly singular). Next natu-

ral step is to construct the topological twists of the *general N = (2, 2)* sigma model. Some important observations in this direction have been presented in [15]. Hopefully better understanding of D-branes and topological twists of the *general N = (2, 2)* sigma model will bring some light to a mirror symmetry beyond the Calabi-Yau case (for the related discussion of mirror symmetry beyond Calabi-Yau manifolds see [11, 5, 9]). We plan to come back to this problem elsewhere.

References

[1] C. Albertsson, U. Lindström and M. Zabzine, "N = 1 supersymmetric sigma model with boundaries. I," Commun. Math. Phys. **233** (2003) 403 [arXiv:hep-th/0111161].

[2] C. Albertsson, U. Lindström and M. Zabzine, "N = 1 supersymmetric sigma model with boundaries. II," Nucl. Phys. B **678** (2004) 295 [arXiv:hep-th/0202069].

[3] A. Y. Alekseev and V. Schomerus, "D-branes in the WZW model," Phys. Rev. D **60** (1999) 061901 [arXiv:hep-th/9812193].

[4] O. Ben-Bassat and M. Boyarchenko, "Submanifolds of generalized complex manifolds," arXiv:math.DG/0309013.

[5] O. Ben-Bassat, "Mirror Symmetry and Generalized Complex Manifolds," arXiv:math.AG/0405303.

[6] F. Bonechi and M. Zabzine, "Poisson sigma model over group manifolds," arXiv:hep-th/0311213.

[7] A. S. Cattaneo and G. Felder, "Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model," arXiv:math.qa/0309180.

[8] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, "The geometry of WZW branes," J. Geom. Phys. **34** (2000) 162 [arXiv:hep-th/9909030].

[9] S. Fidanza, R. Minasian and A. Tomasiello, "Mirror symmetric SU(3)-structure manifolds with NS fluxes," arXiv:hep-th/0311122.

[10] S. J. Gates, C. M. Hull and M. Roček, "Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models," Nucl. Phys. B **248** (1984) 157.

[11] M. Gualtieri, "Generalized complex geometry," Oxford University DPhil thesis, arXiv:math.DG/0401221.

[12] N. Hitchin, "Generalized Calabi-Yau manifolds," Q. J. Math. **54** (2003), no. 3, 281–308, arXiv:math.DG/0209099.

[13] A. Kapustin and D. Orlov, "Vertex algebras, mirror symmetry, and D-branes: The case of complex tori," Commun. Math. Phys. **233** (2003) 79 [arXiv:hep-th/0010293].

[14] A. Kapustin and D. Orlov, "Remarks on A-branes, mirror symmetry, and the Fukaya category," arXiv:hep-th/0109098.

[15] A. Kapustin, "Topological strings on noncommutative manifolds," arXiv:hep-th/0310057.

[16] U. Lindström and M. Zabzine, "N = 2 boundary conditions for non-linear sigma models and Landau-Ginzburg models," JHEP **0302** (2003) 006 [arXiv:hep-th/0209098].

[17] U. Lindström and M. Zabzine, "D-branes in N = 2 WZW models," Phys. Lett. B **560** (2003) 108 [arXiv:hep-th/0212042].

[18] U. Lindström, R. Minasian, A. Tomasiello and M. Zabzine, "Generalized complex manifolds and supersymmetry," arXiv:hep-th/0405085.

[19] S. Lyakhovich and M. Zabzine, "Poisson geometry of sigma models with extended supersymmetry," Phys. Lett. B **548** (2002) 243 [arXiv:hep-th/0210043].

[20] H. Ooguri, Y. Oz and Z. Yin, "D-branes on Calabi-Yau spaces and their mirrors," Nucl. Phys. B **477** (1996) 407 [arXiv:hep-th/9606112].

[21] S. Stanciu, "D-branes in group manifolds," JHEP **0001** (2000) 025 [arXiv:hep-th/9909163].